Effect of Low-Frequency Modulation on Thermal Convection Instability

P. K. Bhatia and B. S. Bhadauria
Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur, India
Reprint requests to Prof. P. K. B.; E-mail: bsbhadauria@rediffmail.com

Z. Naturforsch. **56a,** 509–522 (2001); received February 26, 2001

The stability of a horizontal layer of fluid heated from below is examined when, in addition to a steady temperature difference between the horizontal walls of the layer a time-dependent low-frequency perturbation is applied to the wall temperatures. An asymptotic solution is obtained which describes the behaviour of infinitesimal disturbances to this configuration. Possible stability criteria are analyzed and the results are compared with the known experimental as well as numerical results.

Key words: Thermal Convection; Modulation; Rayleigh Number; Instability.

1. Introduction

The present paper is concerned with the stability of a fluid layer confined between two horizontal planes and heated periodically from below and above. Lord Rayleigh [1] made the first theoretical analysis of the so-called Benard problem concerning the stability of a fluid layer in the presence of a temperature gradient parallel to the gravitational force. Chandrasekhar [2] has given an extensive treatment on this subject based on linear theory. Closely allied mathematically to the present case is the problem of circular couette flow, i.e. the flow between two co-axially rotating vertical cylinders (Taylor instability) when the inner cylinder has a velocity which varies periodically with time, while the outer cylinder is at rest. Donnelly [3] has investigated experimentally the circular couette flow and found that the onset of instability is delayed by the modulation of the angular speed of the inner cylinder with the degree of stabilization rising from zero at high frequency to a maximum at a frequency of 0.274 (ν/d^2), where d is the gap between the cylinders and ν the kinematic viscosity.

Since the problems of Taylor stability and Benard stability are very similar, Venezian [4] has worked out the thermal analogue of Donnelly's experiment and compared the results with the results of Donnelly. Venezian's theory does not find any such finite frequency, as obtained by Donnelly, but finds that for the case of modulation only at the lower surface, the modulation would be stabilizing with maximum stabilization occurring as the frequency goes to zero. However, he suggested that the linear theory ceases to be applicable when the frequency of modulation is sufficiently small.

Rosenblat and Herbert [5] have investigated the linear stability problem for low-frequency modulation and found an asymptotic solution of the problem. Here the periodicity and amplitude criterion were employed to calculate the critical Rayleigh number. In both analyses free-free boundary conditions were used. Rosenblat and Tanaka [6] have used the Galerkin's procedure to solve the linear problem for the more realistic rigid wall boundary conditions. A similar problem has been considered earlier by Gershuni and Zhukhovitskii [7] for a rectangular temperature profile.

Gresho and Sani [8] have treated the linear stability problem with rigid boundaries and found that gravitational modulation can significantly affect the stability limits of the system. Finucane and Kelly [9] have carried out an analytical-experimental investigation to confirm the results of Rosenblat and Herbert. Besides investigating the linear stability, Roppo et al. [10] have also carried out the weakly non-linear analysis of the problem. A numerical solution of the linear Rayleigh-Benard convection was obtained by Weimin and Charles [11] and compared with the analytic solution. On linear basis, Kelly and Hu [12] have investigated the onset of thermal convection in the presence of an oscillatory, non-planar shear flow. Recently Aniss et al. [13] have studied a linear problem of the convection parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and subjected to vertical periodic motion. In their asymptotic analysis they have investigated the influence of gravitational modulation on the instability threshold. More recently we [14] have investigated the Venezian [4] model for more general boundary conditions and compared the results with the results of Venezian.

0932-0784 / 01 / 0600-0509 $ 06.00 © Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com

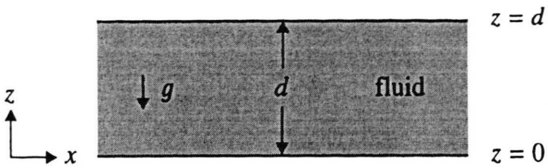

Fig. 1. Benard configuration.

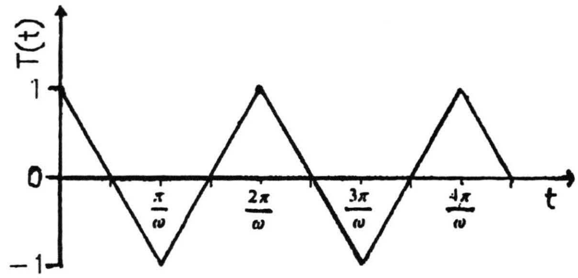

Fig. 2. Saw-tooth variation of T with time t.

The object of the present study is to find an asymptotic solution of the linear stability problem in the case of a very small frequency. Here a saw-tooth and a step function profile are used to modulate the temperature of both the boundaries. The temperature profiles have been expressed in Fourier series. The results of these profiles are compared with experimental results as well as with the theoretical results of Rosenblat and Herbert [5].

2. Formulation

Consider a fluid layer of a viscous, incompressible fluid, confined between two parallel horizontal stress-free planes, a distance d apart. The system is horizontally of infinite extent. The configuration is shown in Figure 1.

The governing equations in the Boussinesq approximation are

$$\frac{\partial V}{\partial t} + V.\nabla V = -\frac{1}{\rho_m}\nabla p + [1-\alpha(T-T_m)]\,X + \nu\nabla^2 V, \quad (2.1)$$

$$\nabla .V = 0, \quad (2.2)$$

$$\frac{\partial T}{\partial t} + V.\nabla T = \kappa\,\nabla^2 T, \quad (2.3)$$

where ρ_m, T_m are constant averages of density and temperature, respectively, $X = (0, 0, -g)$ is the acceleration due to gravity, ν, the kinematic viscosity, κ the thermal diffusivity and α the coefficient of volume expansion, and $V = (u, v, w)$ is the fluid velocity. The relation between ρ_m and T_m is given by

$$\rho = \rho_m[1-\alpha(T-T_m)]. \quad (2.4)$$

The above equations permit an equilibrium solution in which $V = 0$, $T = \bar{T}(z, t)$ is a solution of

$$\frac{\partial \bar{T}}{\partial t} = \kappa\frac{\partial^2 \bar{T}}{\partial z^2}, \quad (2.5)$$

and the pressure $\bar{p}(z, t)$ balances the buoyancy force. To write the boundary conditions, we consider a temperature profile as shown in the Fig. 2 and given by

$$T(t) = \begin{cases} 1+\dfrac{2\omega t}{\pi} & \text{for} \quad \dfrac{-\pi}{\omega} \le t \le 0, \\ 1-\dfrac{2\omega t}{\pi} & \text{for} \quad 0 \le t \le \dfrac{\pi}{\omega}, \end{cases}$$

where ω is the frequency and $\dfrac{2\pi}{\omega}$ is the period of oscillation. The Fourier series of the above function is

$$T(t) = \sum_{m=1}^{\infty} a_m \cos m\omega t, \quad (2.6)$$

where

$$a_m = \frac{4}{m^2\pi^2}(1-\cos m\pi). \quad (2.7)$$

Now we write the externally imposed wall temperatures as follows:

i) when the temperature of the lower boundary as well as of the upper boundary is modulated, we have

$$T(t) = \beta d\left[1+\varepsilon\sum_{m=1}^{\infty} a_m \cos m\omega t\right] \quad \text{at } z = 0, \quad (2.8a)$$

$$= \beta d\,\varepsilon\sum_{m=1}^{\infty} a_m \cos(m\omega t+\phi) \quad \text{at } z = d, \quad (2.8b)$$

ii) when the upper boundary is held at constant temperature and only the lower boundary temperature is modulated we have

$$T(t) = \beta d\left[1+\varepsilon\sum_{m=1}^{\infty} a_m \cos m\omega t\right] \quad \text{at } z = 0, \quad (2.9a)$$

$$= 0 \quad \text{at } z = d. \quad (2.9b)$$

Here ε represents a small amplitude, ϕ is phase angle, and β is the thermal gradient. For the above cases (i) and (ii), (2.5) can be solved. We write

$$\bar{T}(z,t) = T_S(z) + \varepsilon T_1(z,t), \tag{2.10}$$

where $T_S(z)$ is the steady temperature field and εT_1 is the oscillating part. Then the solution is

$$T_S(z) = \Delta T(d-z)/d \tag{2.11}$$

and
$$T_1(z,t) = \mathrm{Re}\left[\sum_{m=1}^{\infty} a_m \left\{a(\lambda_m)\, e^{\lambda_m z/d} + a(-\lambda_m)\, e^{-\lambda_m z/d}\right\} e^{im\omega t}\right], \tag{2.12}$$

where

$$a(\lambda_m) = \frac{e^{i\phi} - e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (i)} \tag{2.13}$$

and
$$a(-\lambda_m) = -\frac{e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (ii)}, \tag{2.14}$$

with
$$\lambda_m^2 = im\omega\, d^2/\kappa. \tag{2.15}$$

For the disturbances, we Fourier-analyze in the xy-plane, and substitute into (2.1)–(2.3) the expressions

$$V = V(z,t)\, e^{i(a_x x + a_y y)},$$
$$T = \bar{T} + \theta(z,t)\, e^{i(a_x x + a_y y)}, \tag{2.16a}$$

$$p = \bar{p} + p(z,t)\, e^{i(a_x x + a_y y)} \tag{2.16b}$$

and neglect non-linear terms. For convenience, the entire problem can be reduced to a single equation for w, the vertical component of the velocity:

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial^2 w}{\partial t^2} - (\kappa + \nu)\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \frac{\partial w}{\partial t} + \kappa\nu\left(\frac{\partial^2}{\partial z^2} - a^2\right)^3 w - \alpha g a^2 \frac{\partial \bar{T}}{\partial z} w = 0, \tag{2.17}$$

where

$$a = (a_x^2 + a_y^2)^{1/2} \tag{2.18}$$

is the horizontal wave-number.

To express the quantities involved in (2.17) in dimensionless form, we put

$$z' = z/d,\ a' = a\,d,\ t' = \omega t \tag{2.19a}$$

$$\bar{T}' = \bar{T}/\Delta T,\ w' = (\nu/\alpha\, g a^2 d^4)\, w. \tag{2.19b}$$

Substituting (2.19) into (2.17) and dropping the primes, (2.17) becomes

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial^2 w}{\partial t^2} - \frac{(1+P)}{\omega}\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \frac{\partial w}{\partial t} + \frac{P}{\omega^2}\left(\frac{\partial^2}{\partial z^2} - a^2\right)^3 w - \frac{PRa^2}{\omega^2}\frac{\partial \bar{T}}{\partial z} w = 0, \tag{2.20}$$

where $P = \nu/\kappa$ is the Prandtl number, $R = \alpha g \Delta T d^3/\nu\kappa$ the Rayleigh number, and $\omega^* = \omega d^2/\kappa$ the non-dimensional frequency. In (2.20) the asterisk has been dropped. Free-free boundary conditions are applied in this problem, therefore the conditions to be satisfied by w are

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{at } z = 0 \text{ and } 1. \tag{2.21}$$

The temperature gradient $\partial\bar{T}/\partial z$ in (2.20), obtained from the dimensionless form of (2.10), is

$$\frac{\partial \bar{T}}{\partial z} = -1 + \varepsilon\,\mathrm{Re}\left[\sum_{m=1}^{\infty} a_m \left\{A(\lambda_m)\, e^{\lambda_m z} + A(-\lambda_m)\, e^{-\lambda_m z}\right\} e^{imt}\right], \tag{2.22}$$

where

$$A(\lambda_m) = \lambda_m \frac{e^{i\phi} - e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (i)} \tag{2.23}$$

and
$$A(\lambda_m) = -\lambda_m \frac{e^{-\lambda_m}}{e^{\lambda_m} - e^{-\lambda_m}} \quad \text{for case (ii)} \tag{2.24}$$

with
$$\lambda_m^2 = im\omega. \tag{2.25}$$

For the solution of (2.20) we write

$$w(z,t) = \sum_{m=1}^{\infty} x_m(t) \sin m\pi z, \tag{2.26}$$

each component of which satisfies (2.21). Now substituting this series into (2.20), multiplying by $\sin n\pi z$ ($n = 1, 2, 3, \ldots$) and integrating with respect to z over the interval (0, 1), a system of ordinary differential equations for the functions $x_n(t)$ is obtained, namely

$$\frac{d^2 x_n}{dt^2} + \frac{(1+P)\,q_n}{\omega}\frac{dx_n}{dt} + \frac{P}{\omega^2 q_n}[q_n^3 - Ra^2]\,x_n - \frac{\varepsilon\,Ra^2 P}{\omega^2 q_n}\sum_{j,m=1}^{\infty} a_j\,x_m\,[g_{mnj}\,e^{ijt} + \tilde{g}_{mnj}\,e^{-ijt}] = 0, \tag{2.27}$$

where ~ denotes "conjugate complex",

$$q_n = n^2\pi^2 + a^2, \tag{2.28}$$

and
$$g_{mnj} = -\int_0^1 \left[A(\lambda_j)\,e^{\lambda_j z} + A(-\lambda_j)\,e^{-\lambda_j z}\right] \cdot \sin m\pi z \sin n\pi z \, dz$$

or

$$g_{mnj} = \frac{ij\omega}{2}\left[\frac{1-(-1)^{m-n}e^{i\phi}}{ij\omega + (m-n)^2\pi^2} - \frac{1-(-1)^{m+n}e^{i\phi}}{ij\omega + (m+n)^2\pi^2}\right] \quad \text{for case (i)} \tag{2.29}$$

and

$$g_{mnj} = \frac{ij\omega}{2}\left[\frac{1}{ij\omega + (m-n)^2\pi^2} - \frac{1}{ij\omega + (m+n)^2\pi^2}\right] \quad \text{for case (ii).} \tag{2.30}$$

3. Low-frequency Approximation

Here we shall obtain an approximate solution for $\omega \leq 1$. It is evident that the system (2.27) is singular if $\omega \to 0$, but it is of classical type and amenable to an asymptotic solution by standard WKB techniques. This approach does not limit ε to small values, although there will be a restriction imposed later.

Now, by expanding the coupling coefficients g_{mnj} in powers of ω, the system (2.27) for case (i) can be written as

$$\omega^2\frac{d^2 x_n}{dt^2} + \omega\,(1+P)\,q_n\frac{dx_n}{dt} + \frac{P}{q_n}\left[q_n^3 - Ra^2\left\{1 + \varepsilon\sum_{j=1}^{\infty} a_j\,\{(1-\cos\phi)\cos jt + \sin\phi\sin jt\}\right\}\right] - \frac{\varepsilon\,Ra^2 P}{q_n}\sum_{j,m=1}^{\infty} a_j\,x_m\left[\omega\left\{r_{mnj}^{(1)}\sin jt + r_{mnj}^{(2)}\cos jt\right\} + \omega^2\left\{s_{mnj}^{(1)}\sin jt + s_{mnj}^{(2)}\cos jt\right\} + 0\,(\omega^3)\right] = 0, \tag{3.1}$$

$(n = 1, 2, 3, \ldots)$,

where

$$r_{mnj}^{(1)} = j\begin{cases}\dfrac{1-\cos\phi}{4m^2\pi^2} & (m=n)\\[2ex] \dfrac{1-(-1)^{m+n}\cos\phi}{(m+n)^2\pi^2} - \dfrac{1-(-1)^{m-n}\cos\phi}{(m-n)^2\pi^2} & (m\neq n)\end{cases}, \tag{3.2}$$

$$r_{mnj}^{(2)} = j\sin\phi\begin{cases}\dfrac{-1}{4m^2\pi^2} & (m=n)\\[2ex] \dfrac{(-1)^{m-n}}{(m-n)^2\pi^2} - \dfrac{(-1)^{m+n}}{(m+n)^2\pi^2} & (m\neq n)\end{cases}, \tag{3.3}$$

$$s_{mnj}^{(1)} = j^2\sin\phi\begin{cases}\dfrac{-1}{16m^4\pi^4} & (m=n)\\[2ex] \dfrac{(-1)^{m-n}}{(m-n)^4\pi^4} - \dfrac{(-1)^{m+n}}{(m+n)^4\pi^4} & (m\neq n)\end{cases}, \tag{3.4}$$

$$s_{mnj}^{(2)} = j^2\begin{cases}\dfrac{-(1-\cos\phi)}{16m^4\pi^4} & (m=n)\\[2ex] \dfrac{1-(-1)^{m-n}\cos\phi}{(m-n)^4\pi^4} - \dfrac{1-(-1)^{m+n}\cos\phi}{(m+n)^4\pi^4} & (m\neq n)\end{cases}. \tag{3.5}$$

If we remove the terms of ϕ from (3.1) and (3.2)–(3.5), the above system reduces for case (ii). In (3.1) we substitute the asymptotic expansion

$$x_n(t,\omega) = \eta_n e^{y(t)/\omega}\{F_n(t) + \omega G_n(t) + \omega^2 H_n(t) + \ldots\} \tag{3.6}$$

(where the η_n are constants representing the initial values of the x_n) and equate to zero coefficients of like powers of ω. The systems corresponding to ω^0, ω^1 and ω^2 are found to be, respectively,

$$F_n \cdot Z_n = 0, \tag{3.7}$$

$$G_n \cdot Z_n = -\left[2y' F_n' + (1+P)\, q_n\, F_n' + y'' F_n'\right] + \frac{\varepsilon\, Ra^2 P}{q_n} \sum_{j,m=1}^{\infty} a_j\, F_m \left\{r_{mnj}^{(1)} \sin jt + r_{mnj}^{(2)} \cos jt\right\}, \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log F_1) = \frac{B}{4}\left[\frac{\sum_{j=1}^{\infty} a_j \left\{r_{11j}^{(1)} \sin jt + r_{11j}^{(2)} \cos jt\right\}}{\left[A + B\sum_{j=1}^{\infty} a_j \{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}\right]^{\frac{1}{2}}} + \frac{\sum_{j=1}^{\infty} j\, a_j \{(1-\cos\phi)\sin jt - \sin\phi\cos jt\}}{A + B\sum_{j=1}^{\infty} a_j \{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}}\right]. \tag{3.14}$$

$$H_n \cdot Z_n = -\left[2y' G_n' + (1+P)\, q_n\, G_n' + y'' G_n'\right] - F_n'' + \frac{\varepsilon\, Ra^2 P}{q_n} \sum_{j,m=1}^{\infty} a_j \cdot \left[G_m \left\{r_{mnj}^{(1)} \sin jt + r_{mnj}^{(2)} \cos jt\right\} + F_m \left\{s_{mnj}^{(1)} \sin jt + s_{mnj}^{(2)} \cos jt\right\}\right], \tag{3.9}$$

where

$$Z_n = y'^2 + (1+P)\, q_n y' + \frac{P}{q_n}\left[q_n^3 - Ra^2 \cdot \left\{1 + \varepsilon \sum_{j=1}^{\infty} a_j \{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}\right\}\right]. \tag{3.10}$$

For the solution of the above system we have from (3.7) either $F_n = 0$ or $Z_n = 0$, and if the latter holds we get a quadratic equation for $y'(t)$, with two roots for each n. However it is easy to show that the largest growth-rate is determined by the greater of the roots when $n = 1$. In particular, when this mode is marginally stable, all others are damped. However it is sufficient to take

$$Z_1 = 0,\ F_2, = F_3 = \ldots = 0, \tag{3.11}$$

and then we have

$$y^1(t) = -\frac{1}{2}(1+P)\, q_n + \frac{1}{2} \cdot \left[A + B \sum_{j=1}^{\infty} a_j \{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}\right]^{\frac{1}{2}}, \tag{3.12}$$

where

$$A = (1-P)^2 q_1^2 + \frac{4Ra^2 P}{q_1},\ B = \varepsilon\frac{4Ra^2 P}{q_1}. \tag{3.13}$$

From (3.8) we find an experession for F_1 as (3.14).

Also, when $n \neq 1$ we have $Z_n \neq 0$, then we get from (3.8) an expression for G_n, $n > 1$ as

$$G_n = \frac{1}{4} B \frac{q_1 F_1}{q_n z_n} \sum_{j=1}^{\infty} a_j \left\{r_{inj}^{(1)} \sin jt + r_{inj}^{(2)} \cos jt\right\}, \quad (n = 1, 2, 3 \ldots). \tag{3.15}$$

To determine G_1, we use (3.11) in (3.9) and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{G_1}{F_1}\right) = \frac{1}{AB(t)}\left[\frac{-F_1''}{F_1} + \frac{1}{4} B \sum_{j=1}^{\infty} a_j \left\{s_{11j}^{(1)} \sin jt + s_{11j}^{(2)} \cos jt\right\} + \frac{B^2}{16} q_1 \sum_{\substack{j=1 \\ m=2}}^{\infty} \frac{a_j^2}{q_m z_m} \left\{r_{m1j}^{(1)} \sin jt + r_{m1j}^{(2)} \cos jt\right\}^2\right], \tag{3.16a}$$

where

$$AB(t) = \left[A + B \sum_{j=1}^{\infty} a_j \cdot \{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}\right]^{\frac{1}{2}}. \tag{3.16b}$$

This solution procedure can be continued and provides an asymptotic approximation for the fastest-growing mode, namely

$$x_1 \sim \eta_1\, e^{y/\omega}\, \{F_1 + \omega\, G_1 + \omega^2 H_1 + \dots\}, \qquad (3.17)$$

where y, F_1 and G_1 are given by (3.12), (3.14) and (3.16), respectively. The solution is subject to the restriction, always present in the WKB method, that the coefficient of the dependent variable in the normalized form of the differential equation should be free from zeros. Therefore from (3.14) and (3.16) we require

$$A + B \sum_{j=1}^{\infty} a_j \,\{(1-\cos\phi)\cos jt + \sin\phi \sin jt\} > 0 \quad (3.18)$$

in the interval of interest, which is $(0,2\pi)$. Hence the condition (3.18) is equivalent to

$$A > 0, \qquad (3.19a)$$

$$A > 2B, \qquad (3.19b)$$

for case (i) and $\quad A > B \quad$ for case (ii). $\qquad$ (3.20)

The condition (3.19a) is valid for all values of ε. Also, if $P = 1$, then from (3.13) the relations (3.19b) and (3.20) reduce to $\varepsilon < 1/2$ and $\varepsilon < 1$, respectively. This puts a restriction on the range of ε. But if $P \neq 1$, the value of ε may be greater than unity, so long as (3.19b) and (3.20) are satisfied.

The expression (3.17) is merely a representation of a solution to the differential equation (3.1), and no conditions of stability have yet been introduced. It is well known (cf. Conrad and Criminale [15], Rosenblat [16]) that alternative stability criterions are possible when the equilibrium state is time-periodic. In the next two sections we examine the consequences of applying two different criteria to the solution.

4. Periodicity Criterion

This is a natural criterion to use in discussing the stability of time-dependent systems and is the one used by Venezian [4] for the present configuration. The object is to determine the value R_p, say, of the Rayleigh number R for which the disturbance x_1 is periodic with period 2π. Since x_1 is the least damped of the modes, and since (3.1) is a system with periodic coefficients to which the standard Floquet theory is applicable, we are assured that all disturbances will decay when $R < R_p$, while at least x_1 will grow when $R > R_p$. We can write this condition on x_1 in the form

$$\log \left| \frac{x_1(2\pi)}{x_1(0)} \right| = 0. \qquad (4.1)$$

Applying the above condition on (3.17), we obtain to order ω^2

$$y(2\pi) - y(0) + \omega^2 \left[\frac{G_1}{F_1}(2\pi) - \frac{G_1}{F_1}(0) \right] = 0. \qquad (4.2)$$

Here y and G_1/F_1 are given by (3.12) and (3.16), respectively. Equation (4.2) contains no term of order ω because its coefficient, $\log F_1$ is identically periodic. Now we write

$$R = R_p = R_p^{(0)} + \omega^2 R_p^{(1)} + \dots \qquad (4.3)$$

and corresponding to this

$$A = A_0 + \omega^2 A_1 + \dots, \quad B = B_0 + \omega^2 B_1 + \dots, \quad (4.4)$$

where, from (3.13) we have

$$A_0 = (1-P)^2 q_1^2 + \frac{4a^2 P R_p^{(0)}}{q_1},\; B_0 = \varepsilon \frac{4a^2 P R_p^{(0)}}{q_1}, \qquad (4.5)$$

$$A_1 = \frac{4a^2 P R_p^{(1)}}{q_1}, \quad B_1 = \varepsilon \frac{4a^2 P R_p^{(1)}}{q_1} = \varepsilon A_1. \quad (4.6)$$

We insert these forms in the above mentioned expressions for y and G_1/F_1 in (4.2). Then by equating the like powers of ω, we obtain, to order ω^2,

$$-\pi(1+P)\, q_1 + \frac{1}{2}\int_0^{2\pi} \left[A_0 + B_0 \sum_{j=1}^{\infty} a_j \,\{(1-\cos\phi)\cos jt + \sin\phi \sin jt\} \right] \mathrm{d}t = 0, \qquad (4.7)$$

$$\frac{1}{4}\int_0^{2\pi} \frac{A_1 + B_1 \sum_{j=1}^{\infty} a_j \,\{(1-\cos\phi)\cos jt + \sin\phi \sin jt\}}{\left[A_0 + B_0 \sum_{j=1}^{\infty} a_j \,\{(1-\cos\phi)\cos jt + \sin\phi \sin jt\} \right]^{\frac{1}{2}}} \,\mathrm{d}t + \left[\frac{G_1}{F_1}(2\pi) - \frac{G_1}{F_1}(0) \right]_{\omega=0} = 0. \qquad (4.8)$$

From (4.7) and (4.8) we can determine $R_p^{(0)}$ and $R_p^{(1)}$, respectively, in terms of other parameters. The values of $R_p^{(0)}$ and $R_p^{(1)}$ will be obtained for the following threes cases: (a) when the boundary temperatures are modulated in phase, i.e. $\phi = 0$; (b) when the modulation is out of phase, i.e. $\phi = \pi$, and (c) when only the temperature of the lower boundary is modulated, the upper boundary being held at a fixed constant temperature. The last case can be recovered by not considering $\cos\phi$ and $\sin\phi$ terms.

Case (a) [$\phi = 0$]; we take $P = 1$, then (4.7) gives

$$R_p^{(0)} = R_c, \tag{4.9}$$

where

$$R_c = q_1^3 / a^2 \tag{4.10a}$$

is the critical Rayleigh number of the classical Benard problem, whose critical wave number is given by

$$a^2 = \frac{\pi^2}{2}. \tag{4.10b}$$

Also, from (4.8) the second order correction to R_p is

$$R_p^{(1)} = \frac{-9\,\varepsilon^2 A_0^2\, C_3}{4\pi^2} \sum_{m=2}^{\infty} \frac{m^2}{q_m\, z_m} \frac{\left[1+(-1)^m\right]^2}{(m^2-1)^4}, \tag{4.11a}$$

where

$$q_m\, z_m = q_m\left[(y')^2 + 2y' q_m + q_m^2\right] - q_1^3, \tag{4.11b}$$

$$y' = \left(\sqrt{A_0} - 2q_1\right)/2, \tag{4.11c}$$

and $$C_3 = \sum_{j=1}^{\infty} j^2 a_j^2. \tag{4.11d}$$

Case (b) [$\phi = \pi$]; from (4.7) we get

$$\pi(1+P)\, q_1 = 2(A_0 + C\, C_1\, B_0)^{1/2} E, \tag{4.12}$$

where

$$E = \int_0^{\pi/2} \left[1 - K \sum_{j=1}^{\infty} a_j \sin^2 jt\right]^{\frac{1}{2}} \mathrm{d}t \tag{4.13}$$

is the elliptic integral of the second kind, dependent on a parameter K given by

$$K = \frac{2\, C\, B_0}{(A_0 + C\, C_1\, B_0)} \tag{4.14a}$$

with $$C = 2 \text{ and } C_1 = \sum_{j=1}^{\infty} a_j. \tag{4.14b}$$

Here the value of the integral $E = E(K)$ is calculated numerically by using Simpson's 1/3-rule (cf. Sastry [17]). The equations (4.12) and (4.14) can be expressed in the form

$$\pi^2(1+P)^2 = 4\Big[(1-P)^2 + 4P\,(1+\varepsilon\, C\, C_1)\, R_p^{(0)}/R_c\Big] E^2 \tag{4.15}$$

with $$K = \frac{8\,\varepsilon\, C\, P\, R_p^{(0)}/R_c}{(1-P)^2 + 4P(1+\varepsilon\, C\, C_1)\, R_p^{(0)}/R_c}. \tag{4.16}$$

Since these two expression do not contain a explicitly, it follows that $R_p^{(0)}$ and R_c have the same wave-number dependence in the sense that the critical value of a for both of them is given (4.10b). If we take $P = 1$, then (4.15) and (4.16) reduce to

$$\frac{R_p^{(0)}}{R_c} = \frac{\pi^2}{4(1+\varepsilon\, C\, C_1)\, E^2} \tag{4.17}$$

and $$K = \frac{2\,\varepsilon\, C}{1+\varepsilon\, C\, C_1}, \tag{4.18}$$

respectively. Now from (4.8) the value of $R_p^{(1)}$ is given by

$$R_p^{(1)} = \frac{3}{4E} \frac{1}{(1+\varepsilon\, C\, C_1)^{\frac{1}{2}}} \int_0^{2\pi} \frac{1}{\left(1+\varepsilon\, C \sum_{j=1}^{\infty} a_j \cos jt\right)^{\frac{1}{2}}} \cdot \left[\frac{F_1''}{F_1} + \frac{\varepsilon\, C A_0}{64\,\pi^4} \sum_{j=1}^{\infty} j^2 a_j \cos jt - \frac{3\,\varepsilon^2 A_0^2}{2\pi^2} \sum_{\substack{j=1\\ m=2}}^{\infty} \frac{m^2\,[1-(-1)^m]^2}{q_m Z_m\,(m^2-1)^4} j^2 a_j^2 \sin^2 jt\right] \mathrm{d}t, \tag{4.19}$$

where

$$\frac{F_1''}{F_1} = \left(\frac{F_1'}{F_1}\right)^2 + \frac{\varepsilon C}{4}\left[\frac{\sqrt{A_0}}{4\pi^2}\frac{\sum\limits_{j=1}^{\infty} j^2 a_j \cos jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)^{\frac{1}{2}}} + \frac{\sum\limits_{j=1}^{\infty} j^2 a_j \cos jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)}\right]$$

$$+ \frac{\varepsilon^2 C^2}{4}\left[\frac{\sqrt{A_0}}{8\pi^2}\frac{\sum\limits_{j=1}^{\infty} j\, a_j \sin jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)^{\frac{3}{2}}} + \frac{\sum\limits_{j=1}^{\infty} j\, a_j \sin jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)^{2}}\right]\left(\sum_{j=1}^{\infty} j\, a_j \sin jt\right), \tag{4.20a}$$

$$\frac{F_1'}{F_1} = \frac{\varepsilon C}{4}\left[\frac{\sqrt{A_0}}{4\pi^2}\frac{\sum\limits_{j=1}^{\infty} j\, a_j \sin jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)^{\frac{1}{2}}} + \frac{\sum\limits_{j=1}^{\infty} j\, a_j \sin jt}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)}\right], \tag{4.20b}$$

$$q_m Z_m = q_m\left[(y')^2 + 2q_m y' + q_m^2\right] - R_p^{(0)} a^2\left(1+\varepsilon C\sum_{j=1}^{\infty} a_j \cos jt\right), \tag{4.20c}$$

and

$$y' = -q_1 + \frac{\sqrt{A_0}}{2}\left(1+\varepsilon C\sum_{j=1}^{\infty} a_j \cos jt\right)^{\frac{1}{2}}. \tag{4.20d}$$

Case (c), when the upper plate is held at a fixed constant temperature, then $C = 1$. If $P = 1$, then the value of $R_p^{(0)}/R_c$ is given by (4.17) with $C = 1$. Also $R_p^{(1)}$ is given by

$$R_p^{(1)} = \frac{3}{4E(1+\varepsilon C C_1)^{\frac{1}{2}}}\int_0^{2\pi}\frac{1}{\left(1+\varepsilon C\sum\limits_{j=1}^{\infty} a_j \cos jt\right)^{\frac{1}{2}}}$$

$$\cdot\left[\frac{F_1''}{F_1} + \frac{\varepsilon C A_0}{64\pi^4}\sum_{j=1}^{\infty} j^2 a_j \cos jt - \frac{3\varepsilon^2 A_0^2}{2\pi^2}\sum_{\substack{j=1\\ m=2}}^{\infty}\frac{m^2}{q_m Z_m (m^2-1)^4}\, j^2 a_j^2 \sin^2 jt\right] dt, \tag{4.21}$$

where $\frac{F_1''}{F_1}, \frac{F_1'}{F_1}, q_m Z_m$ and y' are given by (4.20a), (4.20b), (4.20c) and (4.20d), respectively, for $C = 1$.

Also we consider an another temperature profile, called step function profile, to modulate the wall temperatures (Figure 3). The Fourier series of this function is

$$T(t) = \sum_{m=1}^{\infty} a_m \cos m\omega t,$$

$$a_m = \frac{4}{m\pi}\sin(m\pi/2).$$

Here the values of $R_p^{(0)}$ and $R_p^{(1)}$ for both the temperature profiles, saw-tooth functions as well as step-function, and for all the three cases, have been calculated. The results of case (c) are compared with the Rosenblat and Herbert [5] results. In the Rosenblat and Herbert results, a sinusoidal function is taken for modulating only the lower boundary temperature. We have obtained the results for a sinusoidal function also for cases (a) and (b) and compared them with the above results.

It is easy to verify analytically, for the cases (a) and (b), from (4.15) and (4.16) that the quantity $R_p^{(0)}/R_c$ is

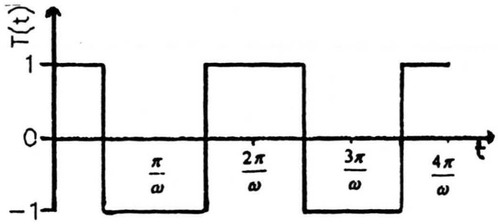

Fig. 3. Step-function variation of temp. T with time t.

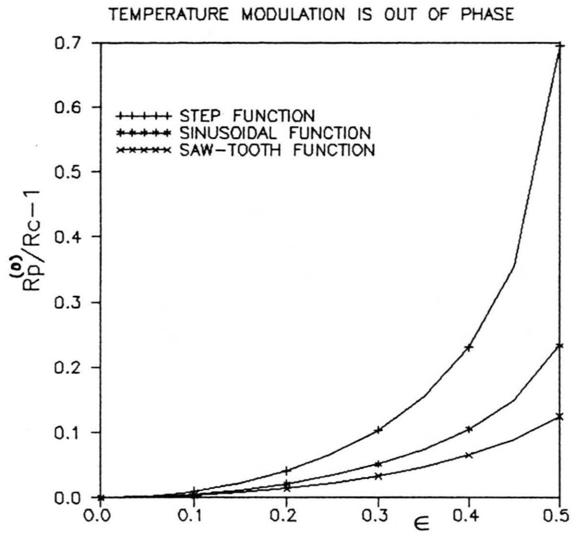

Fig. 4. Periodicity criterion: $P = 1$.

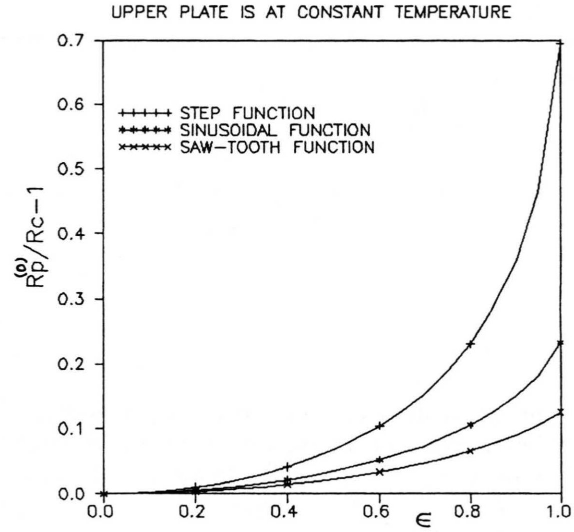

Fig. 5. Periodicity criterion: $P = 1$.

maximum when $P = 1$ (for arbitrary, fixed ε). This means that in the limit $\omega \to 0$ the enhancement of the critical Rayleigh number is greatest when $P = 1$. The actual behaviour of $R_p^{(0)}/R_c$ over a range of values of P can be calculated numerically from (4.15) and (4.16).

Next, we consider the variation of $R_p^{(0)}/R_c$ with ε. For the illustration it is sufficient to take the peak modulation value $P = 1$. In this case $R_p^{(0)}/R_c$ is given by (4.17) and (4.18), with $C_1 = 2$ and 1 for the cases (b) and (c), respectively. The values are now evaluated numerically, and the behaviour of $R_p^{(0)}/R_c$ for the cases (b) and (c) is shown in Figs. 4 and 5, respectively. It is clear from both above figures that the effect of modulation is more stabilizing when the modulation is out of phase than when the upper plate is at constant temperature. For in-phase modulation the value of $R_p^{(0)}$ is given by (4.9) and (4.10); here it is clear that the effect of modulation on the onset of convection is zero.

Finally, the second order correction $R_p^{(1)}$ of R_p is given by (4.11), (4.19) and (4.21) for the cases (a), (b) and (c), respectively. Here the values of $R_p^{(1)}$ for all the three cases is found to be negative. This means that maximum stabilization occurs when $\omega \to 0$ rather than at some finite frequency as observed in the couette flow analogue by Donnelly [3].

It is relevant now to consider whether periodicity is a suitable criterion of stability in the present situation. When $R < R_p$, the solution obtained above is certainly quasi-asymptotically stable. That is any infinitesimal disturbance, whose initial value η_1 is less than some number η, tends to zero as $t \to \infty$. It is, of course, not possible to determine η; it corresponds to the maximum amplitude of those disturbances for which linearization is valid.

It can also be said that disturbances are stable (in the Liapunov sense) for $R \le R_p$, in that we can find a number η_2 such that all disturbances with $\eta_1 < \eta_2$ can be made to remain within prescribed bounds at all times t.

This property, however, is subject to the following qualification. The form of the solution shows that when ω is small and R is close to R_p, the quantity $e^{y/\omega}$ oscillates between very large and very small values. In the limit $\omega \to 0$ this oscillation tends to infinity. It follows that if $|x_1|$ is to have a finite bound, its initial value must be chosen sufficiently small. In particular, η_2 will need to depend on ω, with a variation of the form $e^{-1/\omega}$, and cannot be prescribed independently of ω. Thus, although we can achieve stability, we cannot ensure stability for any disturbance uniformly with respect to ω as $\omega \to 0$.

If, on the other hand, we consider the class of disturbances specified by η, we may have sufficiently substantial growth during some interval of time for the linear theory to break down. In the next section we discuss the stability criterion so as to include the class η within linear theory.

5. Amplitude Criterion

In view of the doubtful validity of the periodicity criterion when used in conjunction with linear theory and low-frequency of modulation, Rosenblat and Herbert [5] have proposed the amplitude criterion, which describes as unstable any disturbance which increases during a cycle by an arbitrary factor of ten. The object of the criterion is to prevent the amplitude of the least stable mode x_1, becoming too large during any part of its cycle. Let the new critical Rayleigh number be R_a. Then obviously $R_a < R_p$, and the mode x_1 will have the form of a damped oscillation. Let $(x_1)_{\min}$ and $(x_1)_{\max}$ be the values of x_1 at a successive minimum and maximum, respectively, occuring at $t = \tau_-$ and $t = \tau_+$ say. Then the disturbance is stable if

$$\log\left[\frac{(x_1)_{\max}}{(x_1)_{\min}}\right] \le M, \tag{5.1}$$

where $M = o(1)$ as $\omega \to 0$. Expression (5.1) gives the required amplitude criterion. The equality sign in (5.1) corresponds to marginal stability. This stability criterion was first used by Rosenblat and Herbert [5] in their study of low-frequency modulation of thermal instability.

From (3.17) we write

$$\frac{(x_1)_{\max}}{(x_1)_{\min}} = \exp\left[(1/\omega)\{y(\tau_+) - y(\tau_-)\}\right] \cdot \left[\frac{F_1(\tau_+)}{F_1(\tau_-)} + o(\omega)\right]. \tag{5.2}$$

Using (5.2) for (5.1), we write

$$\frac{1}{\omega}[y(\tau_+) - y(\tau_-)] + \log\left[\frac{F_1(\tau_+)}{F_1(\tau_-)} + o(\omega)\right] \le M. \tag{5.3}$$

The times $t = \tau$ at which $x_1(t)$ is stationary are given by the zeros of $x_1'(t)$, that is by the equation

$$\left[\frac{y' F_1}{\omega} + y' G_1 + F_1' + o(\omega)\right]_{t=\tau} = 0. \tag{5.4}$$

Now we solve (5.3) and (5.4) simultaneously, and to do this we write an expansion for the critical Rayleigh number in powers of ω, which in this case turn out to be non-integral powers. We put

$$R = R_a = R_a^{(0)} + \omega^N R_a^{(1)} + \dots, \tag{5.5}$$

where N is to be determined. Corresponding to (5.5) we write

$$A = A_0 + \omega^N A_1 + \dots, \quad B = B_0 + \omega^N B_1 + \dots, \tag{5.6}$$

where the A's and B's are as defined in (4.5) and (4.6), except that $R_p^{(0)}$, $R_p^{(1)}$ are replaced by $R_a^{(0)}$, $R_a^{(1)}$, respectively.

Also the consecutive stationary times τ_- and τ_+ have the following properties. If $t = 0$ is taken as an arbitrary reference point, then τ_- and τ_+ coincide with $t = 0$ when $\omega \to 0$. This is apparent from consideration of (5.3) and (5.4) in the limit $\omega \to 0$ as

$$y(\tau_+) = y(\tau_-), \quad (y')_{t=\tau} = 0. \tag{5.7}$$

By circumventing some algebra, here we may write directly

$$\tau = 0 + \omega^{\frac{N}{2}} \tau_1 + \dots. \tag{5.8}$$

Now we introduce (5.5), (5.6) and (5.8) into (5.3) and (5.4) and equate the coefficients of like powers of ω. First, from (5.4) the two leading equations, corresponding to the powers ω^0 and ω^N, respectively, are found to be

$$-\frac{1}{2}(1+P)\, q_1 + \frac{1}{2}\left[A_0 + B_0(1-\cos\phi)\sum_{j=1}^{\infty} a_j\right]^{\frac{1}{2}} = 0, \tag{5.9}$$

$$A_1 + B_1(1-\cos\phi)\sum_{j=1}^{\infty} a_j - \frac{1}{2} B_0 (1-\cos\phi)\, \tau_1^2 \sum_{j=1}^{\infty} j^2 a_j = 0. \tag{5.10}$$

Then from (5.9) we get immediately

$$R_a^{(0)} = \frac{R_c}{1 + \varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} a_j}. \tag{5.11}$$

Using (4.6) in (5.10), we get

$$\tau_1 = \pm\left[\frac{2A_1\left\{1+\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} a_j\right\}}{B_0(1-\cos\phi)\sum_{j=1}^{\infty} j^2 a_j}\right]^{\frac{1}{2}}$$

or

$$\tau_1 = \pm\left[\frac{2\left\{1+\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} a_j\right\} R_a^{(1)}}{\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} j^2 a_j R_a^{(0)}}\right]^{\frac{1}{2}}. \tag{5.12}$$

From (5.3), for marginal stability we can write

$$\int_{\tau_-}^{\tau_+} y'(s)\,\mathrm{d}s + o(\omega^2) = \omega M. \tag{5.13}$$

From (3.12) and the preceding results we can easily show that

$$\int_{\tau_-}^{\tau_+} y'(s)\,\mathrm{d}s = \omega^{\frac{3}{2}N}\left(\frac{R_a^{(1)}}{R_a^{(0)}}\right)^{\frac{3}{2}} \cdot \frac{2^{5/2}}{3}\,\frac{P q_1}{1+P}\left[\frac{1+\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} a_j}{\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} j^2 a_j}\right]^{\frac{1}{2}}. \tag{5.14}$$

Substitution of (5.14) into (5.13) gives

$$N = \frac{2}{3} \tag{5.15}$$

and

$$\frac{R_a^{(1)}}{R_a^{(0)}} = \frac{1}{2}\left[\frac{3M(1+P)}{2P q_1}\right]^{\frac{2}{3}}\left[\frac{\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} j^2 a_j}{1+\varepsilon(1-\cos\phi)\sum_{j=1}^{\infty} a_j}\right]^{\frac{1}{3}}. \tag{5.16}$$

Finally we introduce (5.11), (5.15) and (5.16) into (5.5). This now becomes

$$R_a = \frac{R_c}{1+\varepsilon(1-\cos\phi)\,C_1}\left[1+\omega^{\frac{2}{3}}\mu+\ldots\right], \tag{5.17}$$

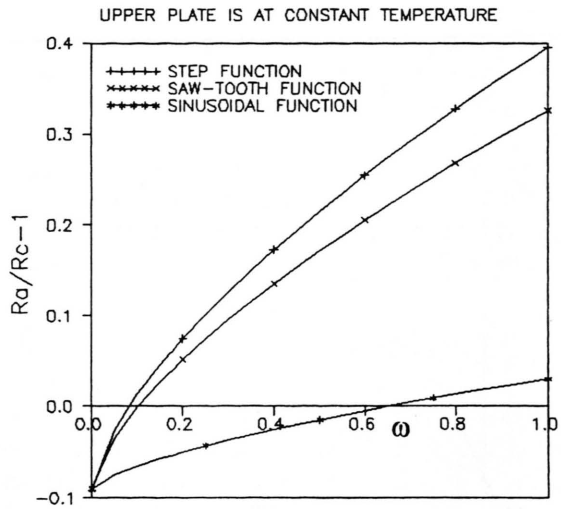

Fig. 6. Amplitude criterion: $\varepsilon = 0.1$.

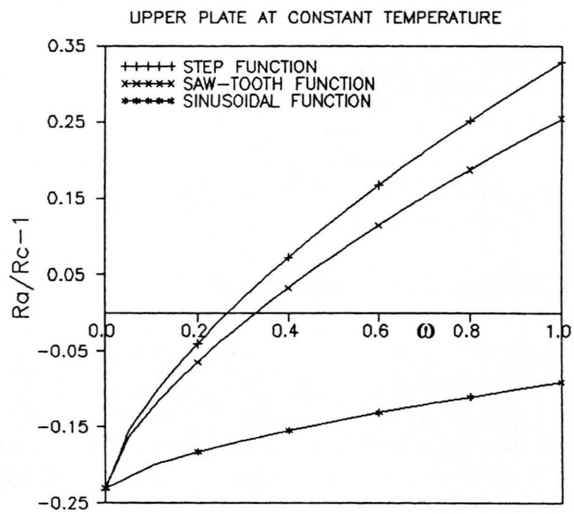

Fig. 7. Amplitude criterion: $\varepsilon = 0.3$.

where

$$\mu = \frac{1}{2}\left[\frac{3M(1+P)}{2P q_1}\right]^{\frac{2}{3}}\left[\frac{\varepsilon(1-\cos\phi)\,C_2}{1+\varepsilon(1-\cos\phi)\,C_1}\right]^{\frac{1}{3}}, \tag{5.18}$$

$C_2 = \sum_{j=1}^{\infty} j^2 a_j$, and R_c and C_1 are given by (4.10a) and (4.14b), respectively.

Equation (5.17) gives the value of the critical Rayleigh number according to the amplitude criterion. The

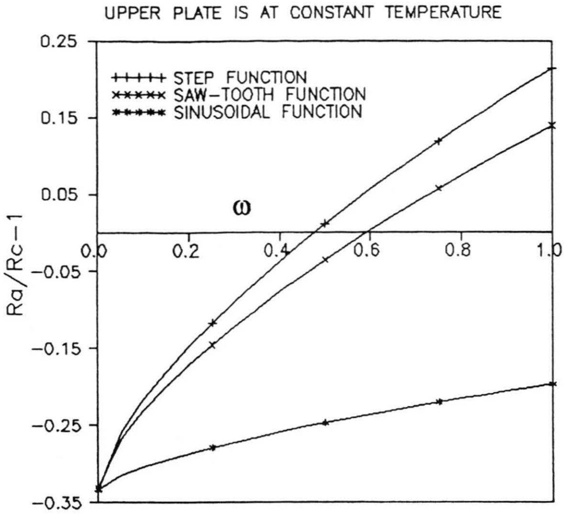

Fig. 8. Amplitude criterion: $\varepsilon = 0.5$.

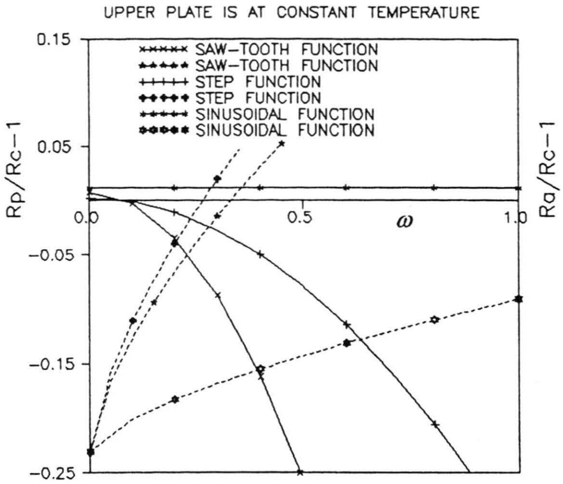

Fig. 10. Variation of R_a/R_c and R_p/R_c with ω, $\varepsilon = 0.3$.

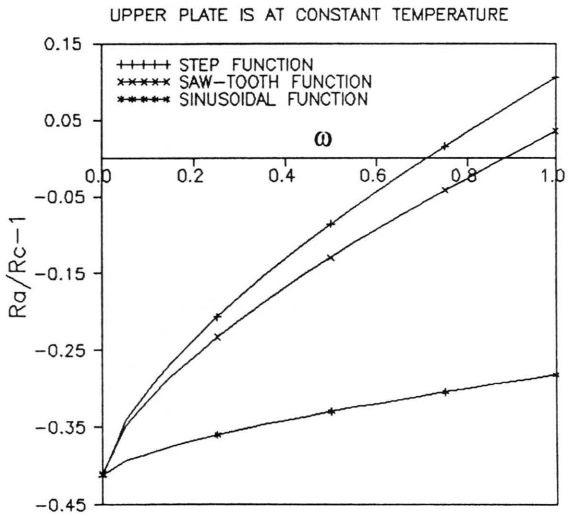

Fig. 9. Periodicity criterion: $\varepsilon = 0.7$.

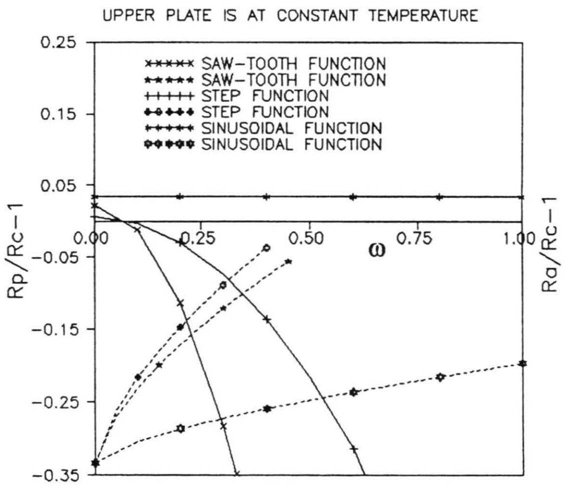

Fig. 11. Variation of R_p/R_c and R_a/R_c with ω, $\varepsilon = 0.5$.

value of R_a is indeterminate to the extent that the constant M is not specified within the criterion. The qualitative trend, however, is clear: the Rayleigh number increases with increasing ω, commencing with the quasi-steady value at $\omega = 0$. It is also apparent from (5.17) that the critical wave-number is in this case subject to a second order modification of order $\omega^{2/3}$. The exact form again depends on the undetermined constant M.

The nature of the solution (5.17), for case (c) is illustrated in the Figures 6–9. For definiteness we have chosen $M = 2.25$, which corresponds to a ten-fold amplification of the disturbance. The figures show the variation of R_a with ω in the range $0 < \omega < 1$, for different values of ε with $P = 1$. Also here the results for different temperature profiles have been compared. For a sinusoidal profile, our results are different from those obtained by Rosenblat and Herbert [5]. This would be because they might have not considered the factor 2 which is under an integral power and in the denominator of (5.18). For case (b), the same graphs are shown in Figs. 6–9, but

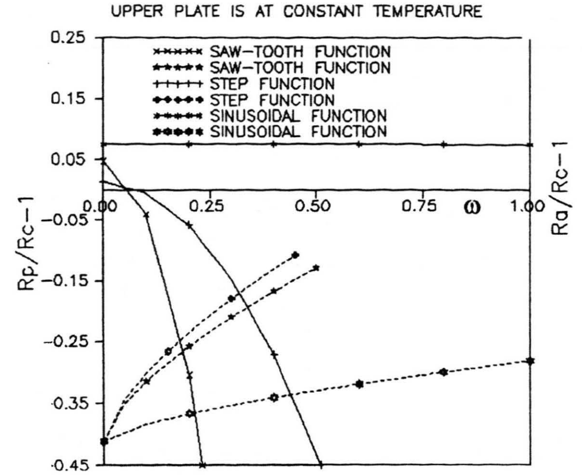

Fig. 12. Variation of R_p/R_c and R_a/R_c with ω, $\varepsilon = 0.7$.

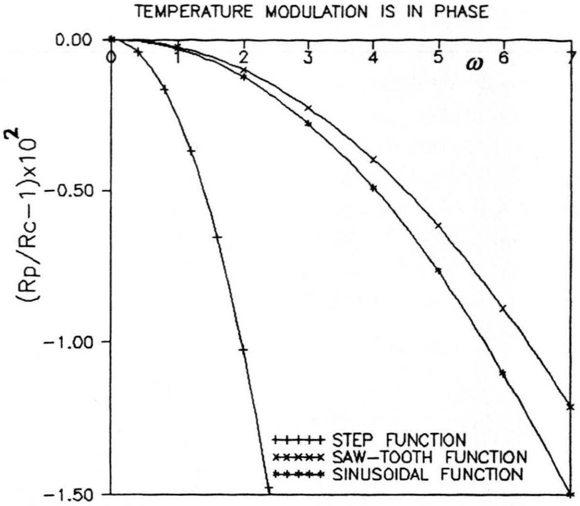

Fig. 14. Variation of R_p/R_c with ω, $\varepsilon = 0.7$.

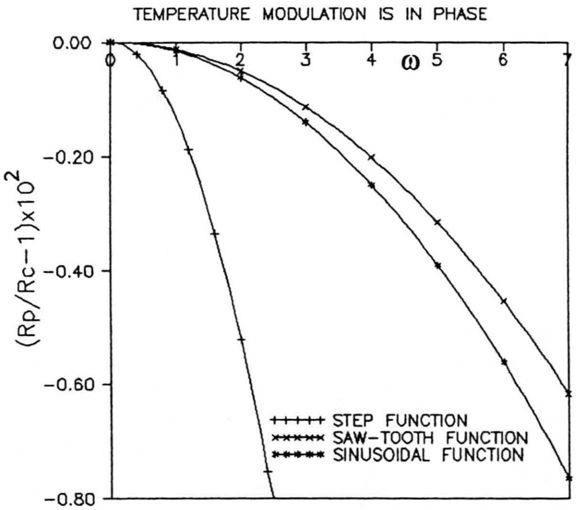

Fig. 13. Variation of R_p/R_c with ω, $\varepsilon = 0.5$.

here the values of R_a are double of those obtained in case (c). For different temperature profiles and at different values of ε, the above figures give the upper bound of the Rayleigh number, for a class of disturbances to be stable (on linear theory).

6. Discussion

The discussion of the preceding two sections can be summarized as follows: for a class of disturbances (those for which linearization is valid and whose initial values are independent of the frequency) the periodicity criterion is a sufficient condition for instability, in that the inequality

$$R > R_p \tag{6.1}$$

ensures asymptotic growth of the disturbances according to linear theory.

On the other hand, the condition

$$R < R_a \tag{6.2}$$

can be regarded as a sufficient condition for stability (on linear theory) for the same class of disturbances, in the sense that their magnitude at any instead can be a priory restricted.

In the case

$$R_a < R < R_p \tag{6.3}$$

our solutions predict that these disturbances decay as $t \to \infty$, but have extreme oscillations at finite t for $\omega \to 0$. It is not possible to predict the actual behaviour of such disturbances without a non-linear analysis. That is, we cannot say whether a non-linear effect tends to stabilize, by decreasing the amplitude of the oscillation, or to destabilize by reinforcing them. Also, when $R < R_p$ there is a class of disturbances which not only decay as $t \to \infty$ but also have finite bounds on their variation. This class, however, contains only disturbances with frequency-dependent initial values and becomes vanishingly small as $\omega \to 0$.

In Figs. 10–12 we have represented R_p and R_a, on the same diagram. In the figures the solid lines refer to R_p

and the broken lines ot R_a. In each figure, R_p and R_a, intersect at some value of ω, indicating the minimum frequency for which linearization is valid. For a step function profile, saw-tooth profile and sinusoidal profile, respectively these values are found to be $\omega = 0.25$, 0.25, 2.3 for $\varepsilon = 0.3$, $\omega = 0.3$, 0.24, 4.4 for $\varepsilon = 0.5$ and $\omega = 0.3$, 0.18, 7 for $\varepsilon = 0.7$.

Finally, Figs. 13 and 14 are devoted to the variation of R_p with ω, when the wall temperatures are modulated in phase. It is found that the effect of second order correction to R_p for the sinusoidal profile is more destabilizing than for the saw-tooth profile but less destabilizing than for the step function profile.

Acknowledgement

This work was done during the tenure of a major research project awarded by UGC to one of the authors (P. K. Bhatia). The financial assistance to Prof. P. K. Bhatia from UGC is gratefully acknowledged.

[1] Lord Rayleigh, Sci. Papers **6,** 432 (1916).
[2] S. Chandrasekhar, Hydrodynamic, and Hydromagnetic Stability, Oxford University Press, Oxford 1961.
[3] R. J. Donnelly, Proc. Roy. Soc. A **281,** 130 (1964).
[4] G. Venezian, J. Fluid Mech. **35,** 243 (1969).
[5] S. Rosenblat and D. M. Herbert, J. Fluid Mech. **43,** 385 (1970).
[6] S. Rosenblat and G. A. Tanaka, Phys. Fluids **14,** 1319 (1971).
[7] G. Z. Gershuni and E. M. Zhukhovitskii, J. Appl. Math. Mech. **27,** 1197 (1963).
[8] P. M. Gresho and R. L. Sani, J. Fluid Mech. **40,** 783 (1970).
[9] R. G. Finucane and R. E. Kelly, Int. J. Heat Mass Transfer **19,** 71 (1976).
[10] M. N. Roppo, S. H. Davis, and S. Rosenblat, Phys. Fluids **27,** 796 (1984).
[11] Xu Weimin and A. Charles, Lin Tellus **45A,** 193 (1993).
[12] R. E. Kelly and H.-C. Hu, J. Fluid Mech. **249,** 373 (1993).
[13] S. Aniss, M. Souhar, and M. Belhaq, Phys. Fluids **12,** 262 (2000).
[14] P. K. Bhatia and B. S. Bhadauria, Z. Naturforsch. **55a,** 957 (2000).
[15] P. W. Conrad and W. O. Criminale, Z. Angew. Math. Phys. **16,** 233 (1965).
[16] S. Rosenblat, J. Fluid Mech. **33,** 321 (1968).
[17] S. S. Sastry, Introductory Methods of Numerical Analysis, 2nd Ed. Prentice-Hall of India Private Limited, New Delhi 1993.